term

$$\sum_j K_{ji}\,(t_j - t_i) + S_i$$

can be negative. Actually, Eqs. (4) and (6) do not follow from Eqs. (3) and (5) because the term

$$\sum_j K_{ji}(t_j - t_i) + S_i$$

can take on negative values that in turn reverse the directions of the inequalities. Moreover, a test is needed to examine the term

$$\sum_j K_{ji}(t_j - t_i) + S_i$$

for possible zero values. If a value of zero is encountered, either node i must be neglected or the term

$$\sum_j K_{ji}(t_j - t_i) + S_i$$

must be assigned a very small value in order that division by zero is not attempted. Also, Gross has not clarified which node in a system is the "most sensitive node." Is it the node that has the minimum value of time step per Eq. (4) or per Eq. (6)? That is, does one find the node with the minimum time step per Eq. (4) and then apply Eq. (6) to establish the compute time step, or does one apply Eq. (6) to all nodes and use the minimum value as the compute time step?

These questions can be resolved by revising and enlarging Gross's development as follows:

1) Replace Eq. (3) with

$$|t_i^+ - t_i| \leq |t_i^* - t_i| \tag{3a}$$

This results in

$$\Delta\tau_i \leq C_i/\sum_j K_{ji}$$

2) Replace Eq. (5) with

$$|t_i^+ - t_i| \leq |t_i^* - t_i| + |\Delta t| \tag{5a}$$

This results in

$$\Delta\tau_i \leq \frac{C_i}{\sum_j K_{ji}}\left[1 + \frac{\Delta t \sum_j K_{ji}}{\left|\sum_j K_{ji}(t_j - t_i) + S_i\right|}\right] \tag{6a}$$

Equation (6a) is offered as a stability criterion that will force a finite-difference solution to be bounded within an error of Δt. Also, it can be shown that one must apply Eq. (6a) to all nodes and then use the minimum value found as the compute time step. This is proven by considering node k with neighboring nodes l. If node i is taken to be the most sensitive node we write

$$t_k^+ = t_k + \left[1 + \frac{\Delta t \sum_j K_{ji}}{\left|\sum_j K_{ji}(t_j - t_i) + S_i\right|}\right] \frac{C_i/\sum_j K_{ji}}{C_k/\sum_l K_{lk}}\,(t_k^* - t_k) \tag{7}$$

$$t_k^+ = t_k\left(1 - \frac{\Delta\tau_i}{\Delta\tau_k}\right) + \frac{\Delta\tau_i}{\Delta\tau_k}\left[t_k^* + \frac{\Delta t \sum_l K_{lk}(t_l - t_k) + S_k}{\left|\sum_l K_{lk}(t_l - t_k) + S_k\right|}\right] \tag{8}$$

where $\Delta\tau_i$ and $\Delta\tau_k$ are per Eq. (6a).

Equations (7) and (8) are expressions for the "new" temperature at the arbitrary node k. These equations are derived by writing Eq. (1) for node k and substituting Eq. (6a) for $\Delta\tau_i$. Equation (2) is used to bring t_k^* into the equations and then the equations are arranged in terms of ratios of $C/\Sigma K$'s and ratios of $\Delta\tau$'s per Eq. (6a). Relative to Eq. (7), we imagine that node i has a minimum value of $C/\Sigma K$ and we find that t_k^+ is not bounded, since the term

$$\left|\sum_j K_{ji}\,(t_j - t_i) + S_i\right|$$

can take on any value. Relative to Eq. (8), we imagine that node i has a minimum value of the time step defined by Eq. (6a). Here we find that t_k^+ is bounded.

As a final comment, we wonder just how valuable Eq. (6a) is. Even though one can use larger computer time steps and retain stability, does one actually save on computer time relative to the usual attack using Eq. (4)? This question arises since Eq. (6a) must be evaluated for each node at each time step, and this certainly is more time consuming than applying Eq. (4). It is felt, for example, that an increase in computer time is certainly probable for problems that have nodal time constants ($c/\Sigma K$) very nearly equal for all of the nodes.

References

[1] Gross, S., "An improved finite-difference method for heat transfer calculations," J. Spacecraft Rockets **4**, 538 (1967).

Reply by Author to R. K. McMordie and L. W. Mordock

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McMordie and Mordock offer a similar but more rigorous development, since their starting point, Eq. (3a), is an exact representation of the statement preceding Eq. (4) in Ref. 1. The final results, however, are the same. Equation (6a) is equivalent to Eq. (10) of Ref. 1, for Δt may be either positive or negative, and will have the same sign as the term

$$\sum_j K_{ji}(t_j - t_i) + S_i$$

Thus, computing time steps cannot be less than that given by Eq. (4) [Eq. (7) of Ref. 1]. The only advantage that Eq. (10) of Ref. 1 has over Eq. (6a) is that, by keeping track of signs, different criteria could be used for rising and falling temperatures.

McMordie and Mordock are unduly concerned over possible zero values for the term

$$\sum_j K_i(t_{ij} - t_i) + S_i$$

It is common practice to place an upper limit on the time step in order to evaluate subroutines and boundary conditions frequently enough, as well as to assure that calculations are spaced closely enough to adequately describe the results.

Equations (7) and (8) fail to prove that the same computing time step must be applied to all nodes. I have solved problems where the same computing time step was not used for all nodes, and have found, as expected, that this can save machine time.

Concern over more machine time to compute Eqs. (6a) than Eq. (4) is not warranted in most problems. The extra terms in Eq. (6a) which are not in Eq. (4) have already been evaluated for calculating temperature, so relatively few extra computations are required. Thus, computation time for problems is approximately inversely proportional to the computing time step.

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